Reply to "Fractal potentials from energy levels"

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We point out in this reply that Ramani, Grammaticos, and Caurier have made a useful technical improvement in the quantum inversion method we used, but the spectrum they worked with is not suitable for the purpose they claim.

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Ramani, Grammaticos, and Caurier have made a useful technical improvement to our method of fitting a one-dimensional potential to a given fluctuating spectrum [1,2]. They construct their potential by means of the "dressing" transformation, which is a special case of the well-known supersymmetric spectrum transformation [3–5]. This refinement allows them to fit accurately many more levels than in our original work, for the same effort, which in principle should improve on the estimation of the fractal dimension of the resulting potential. Further improvement could be made by following Baye [3,8], who has shown how to add states at an arbitrary point in the spectrum. This would allow one to work from the bottom of the spectrum up, rather than top down.

However, this improvement in the inversion technique is undercut by the choice of the spectrum to be fitted. First of all, they have fitted a *different* spectrum than we did. The only common feature of their spectrum with the Riemann zeroes [6] is the unfolded nearest level spacing. It is not apparent how the average level density and the long range correlations of the spectrum might affect the fractal dimension analysis. Thus to compare the fractal dimension obtained from their calculation with ours is questionable.

Furthermore, they chose an *improper* spectrum. As we know, there are three important aspects of spectral properties to be considered: the average density, long range correlations, and nearest-level spacing distribution. By performing a spectrum unfolding procedure, one standardizes the global feature of the spectrum in order to highlight the fluctuation property. However, it is essential that both the nearest-level spacing distribution and spectral rigidity (Δ_3) be reflected in the chosen spectrum. A spectrum produced by random local perturbation to the harmonic oscillator spectrum has no long range correlation, and thus cannot have the correct Δ_3 statistics for GUE-type systems. In fact, Δ_3 will be that of a Poisson distribution. Without the proper long range correlation, the spectrum they are fitting should appear more "random" than a standard GUE spectrum, and this is possibly the reason that they observe a larger fractal dimension than ours.

Lastly, the potential obtained by inversion of a set of energy levels is not unique. According to the Marchenko inversion method [7], given the phase shifts at all energies one can construct a unique local potential to reproduce them. If there are in addition one or more bound states, the kernel of the Marchenko equations is modified by the addition of terms which depend on b, the square root of the bound state energy, and a constant A_S which is arbitrary. Thus one can add as many bound states as one likes, at arbitrary energies, and each one with an arbitrary A_S attached to it. Neither of us has exploited this degree of freedom, so each of our potentials contains a large degree of arbitrariness. It is not surprising that different potentials result. What is clear is that $d > 1$ and this was the important point that we made.

In summary, Ramani *et al.* have usefully pointed out that more powerful inversion techniques can be applied to the problem at hand. These appear to be necessary to determine the fractal dimension to high accuracy. The question of whether the fractal dimension is uniquely related to the spectrum, or depends upon the inversion technique, remains open. Had they fitted the actual Riemann zeroes, the work would have been more relevant to our paper.

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